Physics Letters A 375 (2011) 1309–1318

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

How complex a dynamical network can be?

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ARTICLE INFO

Article history:
Received 17 March 2009
Received in revised form 9 September 2010
Accepted 31 January 2011
Available online 3 February 2011
Communicated by A.P. Fordy

ABSTRACT

Positive Lyapunov exponents measure the asymptotic exponential divergence of nearby trajectories of a dynamical system. Not only they quantify how chaotic a dynamical system is, but since their sum is an upper bound for the rate of information production, they also provide a convenient way to quantify the complexity of a dynamical network. We conjecture based on numerical evidences that for a large class of dynamical networks composed by equal nodes, the sum of the positive Lyapunov exponents is bounded by the sum of all the positive Lyapunov exponents of both the synchronization manifold and its transversal directions, the last quantity being in principle easier to compute than the latter. As applications of our conjecture we: (i) show that a dynamical network composed of equal nodes and whose nodes are fully linearly connected produces more information than similar networks but whose nodes are connected with any other possible connecting topology; (ii) show how one can calculate upper bounds for the information production of realistic networks whose nodes have parameter mismatches, randomly chosen; (iii) discuss how to predict the behavior of a large dynamical network by knowing the information provided by a system composed of only two coupled nodes.

1. Introduction

The relation between topology (way nodes are connected) and behavior in a dynamical network, networks composed by nodes described by some deterministic dynamics whose equations of motion are known, is a fundamental question whose answer may help understand the collective behavior [1] of a variety of complex systems ranging from particle-like chemical waves [2], light propagation in dielectric structures [3], neural networks [4] and metabolic networks [5].

The work of Kuramoto [6] and the works of Pecora and collaborators [7,8] laid the foundations of a theoretical framework for studying the relation between topology and behavior in dynamical networks. In particular, the latter opened up a new way to study the onset of complete synchronization in dynamical networks [9–11] composed of equal node dynamics.

At the present moment, it is important to understand from a theoretical perspective the relation between connecting topology and behavior in dynamical networks whose nodes are not only far away from complete synchronization (desynchronous) but also nodes that interact among themselves simultaneously by linear and non-linear means.

In order to understand this relationship between connecting topology and behavior, we first quantify behavior of a chaotic network by the sum of all the positive Lyapunov exponents of the network, a quantity that is related to the amount of information produced by the network. Then, we show how one can calculate upper or lower bounds for that sum, in terms of exponents that are functions of the connecting topology of the network.

Information is an important concept [12]. It measures how much uncertainty one has about an event before it happens and it is therefore a measure of how complex a system is. Measuring the information produced by a chaotic systems, i.e. its Shannon's entropy, is extremely difficult because one has to calculate an integral of the probabilities of the trajectory along a chaotic set that is fractal. But, for chaotic systems that have absolutely continuous conditional measures, one can calculate Shannon's entropy per unit of time, a quantity known as Kolmogorov–Sinai (KS) entropy [13], by summing all the positive Lyapunov exponents [14]. A system that has absolutely continuous conditional measures is a system whose trajectory continuously distributes along unstable directions. These systems form a large class of well-known (non-uniformly hyperbolic) systems [15]: the Hénon family; Hénon-like attractor arising from Homoclinic bifurcations; strange attractors

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2011.01.054

arising from Hopf bifurcations (e.g. Rössler oscillator); some classes of mechanical models with periodic forcing.

We are not aware of any rigorous result proving the equivalence of the KS entropy and the sum of Lyapunov exponent for the Hindmarsh–Rose neural model neither to a network constructed with it, as the networks considered in this work. But the chaotic attractors arising in this neuron model are similar to the ones appearing from Homoclinic bifurcations. Additionally, for two coupled neurons, we have numerically shown that a lower bound estimation of the KS entropy is indeed close to the sum of all the positive Lyapunov exponents. Despite the lack of a rigorous proof, we will assume that the results in Refs. [14,15] apply in here in the sense that the sum of the positive Lyapunov exponents provide a good estimation for the KS entropy and therefore this sum is a convenient way to quantify how complex a network is.[3]

Our main result, written in terms of a conjecture, states that we can predict whether the sum of all the positive Lyapunov exponents of a large dynamical network is larger or smaller than a given value that can be analytically or semi-analytically calculated. And therefore, even before simulating a large dynamical network, we can estimate how much information such a network can produce. More rigorously, we conjecture that an UPPER (or LOWER) bound for the sum of the Lyapunov exponents of a dynamical network with some special properties[4] and an arbitrary size, formed by nodes possessing equal dynamics, can be analytically calculated by only using information coming from the behavior of two coupled nodes.

It is often considered that the complexity of a network can be quantified by typical characteristics as the average degree, the network's connecting topology, the minimal and maximal degree, the average or minimal path length connecting two nodes, and others. But these characteristics are a measure of the structure of the network and not of the behavior of it. In this work, at least for the class of networks considered here, we can state that these dynamical networks behave in only two ways, regardless the many characteristics that quantify the network's structure: the behaviors UPPER and LOWER. In other words, if nodes of a dynamical network interact by a coupling function that induces a LOWER (or UPPER) character, this character will not be modified by the use of other connecting topologies or by increasing the number of nodes of the network, as long as the nodes possess equal dynamics.

To justify our conjecture, we use dynamical networks of linear and non-linear maps coupled by linear terms, and neural networks of highly non-linear neurons (Hindmarsh–Rose (HR) neurons [17]) connected by both linear (electrical synapses) and non-linear couplings (chemical synapses).

As applications of our conjecture, we show in Section 9.1 how one can calculate an upper bound for the Kolmogorov–Sinai entropy of a network with equal nodes and whose nodes are fully connected. Then, we show analytically (with numerical verification) that the maximal value of this upper bound (varying the linear coupling strength) is larger than the maximal value of the Kolmogorov–Sinai entropy of UPPER networks whose nodes are linearly connected with any other connecting topology. In Section 9.2 we show that an upper bound for the Kolmogorov–Sinai entropy of an UPPER network with equal nodes is larger than this entropy for an equivalent UPPER network (same connecting topology and number of nodes) but whose nodes have parameter mismatches. Therefore, even though networks with equal nodes might not be realistic, their entropy production is an upper bound for the entropy production of more realistic networks. We finally discuss in Section 9.3 how our conjecture can be used to predict whether a LOWER network formed by nodes that when isolated are chaotic (periodic) will maintain such a chaotic behavior, then predicting how complex larger dynamical networks can be.

2. Lyapunov exponents, conditional Lyapunov exponents, and an introduction to our conjecture

To describe our conjecture, we first need to understand what we mean by conditional Lyapunov exponents. Imagine two equal 1-dimensional systems, X and Y, coupled in a way such that a synchronization manifold exists. The trajectory of this coupled systems is represented by a pair of variables (x_i, y_i), and x_0 and y_0 are the initial conditions of systems X and Y, respectively, being that after T iterations these initial conditions go to the point (x_T, y_T).

Since that a synchronization manifold exists, if the initial conditions lie along the synchronization manifold, i.e. $x_0 = y_0$, they will remain there forever under the action of the system, i.e. $x_T = y_T$. Now, calculate the Lyapunov exponents of a trajectory starting with these equal initial conditions. There are 2 Lyapunov exponents. One exponent gives the information of how much nearby points exponentially diverge along the synchronization manifold. The other exponent gives the information of how much nearby points exponentially diverge along a direction transversal to the synchronization manifold. Since these Lyapunov exponents were measured along the synchronization manifold and the direction transversal to it, we call them conditional exponents. Let us now define the quantity Λ_C as to be the sum of all the positive conditional Lyapunov exponents (along the synchronization manifold and all its transversal directions). Why do we use the word “conditional” in our terminology? Because a trajectory along the synchronization manifold is a special solution of this coupled system. There can be many other asymptotic solutions, each one with their basin of attraction located outside the synchronization manifold. In Fig. 1, the dashed line represent this special solution along the synchronization manifold.

Now, let us assume (for the sake of clarification) that if the system is set with randomly set initial conditions such that $x_0 \neq y_0$, after a transient time, the trajectory goes to only one attractor. We assume the synchronization manifold to be unstable and therefore trajectories starting with initial conditions close to the synchronization manifold eventually go to this attractor. So, we assume that the random initial conditions lie in the basin of attraction of this attractor. So, this initial condition is considered to be a typical initial condition because it leads the trajectory to a unique attractor. This attractor is represented in Fig. 1 by the two large

[3] According to the Ruelle formula, for ergodic differentiable systems on compact spaces, the Kolmogorov–Sinai entropy is bounded above by the sum of the positive Lyapunov exponents of the system. If the systems admits an SRB measure, then the Kolmogorov–Sinai entropy is exactly equal to the sum of the positive Lyapunov exponents of the system [16,15]. For the networks here considered formed by dissipative systems that possess an attractor whose measure is completely supported by an unstable manifold, such an equality should be satisfied. Therefore, a bound for $\sum \lambda_m^+$ implies a bound for H_{KS}. In any case, if it is not certain that such an equality holds, notice that the sum of the positive Lyapunov exponents will be always a measure of entropy production per unit time, since it measures the ratio with which partitions should be created in order to define proper states in a dynamical system. So, it is irrelevant to our conjecture whether the sum of the positive Lyapunov exponents represents the KS entropy. Notice that for the class of dynamical systems as the here considered networks, Ruelle [20] has proved that $H_{KS} \leqslant \sum \lambda_m^+$, and therefore, if the sum of the positive conditional Lyapunov exponents, denoted by Λ_C, is an upper bound for $\sum \lambda_m^+$, i.e., $\Lambda_C \geqslant \sum \lambda_m^+$, then, it implies that $\Lambda_C \geqslant H_{KS}$ since it is always true that $H_{KS} \leqslant \sum \lambda_m^+$. The conditional exponents are the Lyapunov exponents of the synchronization manifold and the Lyapunov exponents along the directions transversal to the Lyapunov exponent.

[4] We consider networks of nodes possessing equal dynamics connected simultaneously by linear and non-linear means. The connecting Laplacian matrix that describes the topology under which the nodes are connected linearly and non-linearly are denoted by $\mathcal{G}$ and $\mathcal{C}$, respectively. The strengths of the linear and non-linear couplings are σ and g respectively. If the nodes in the network are connected by only linear couplings ($g = 0$), $\mathcal{G}$ can be an arbitrary Laplacian matrix. If nodes are simultaneously connected by linear and non-linear couplings then $\mathcal{G}$ and $\mathcal{C}$ must commute and every node must receive the same number k of non-linear connections coming from other nodes.

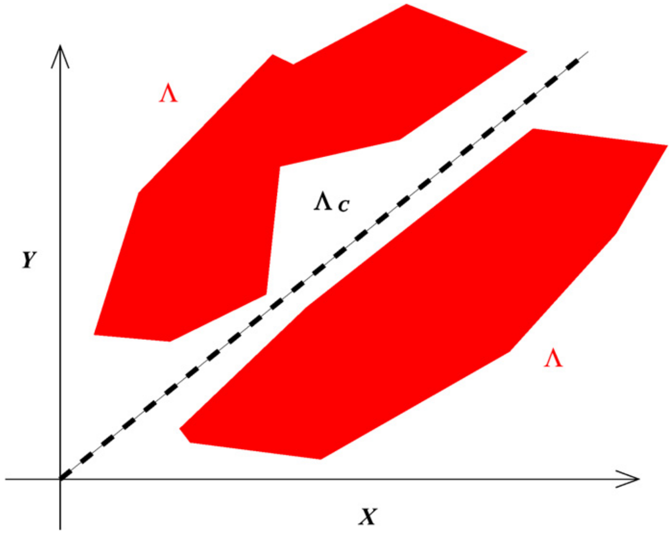

Fig. 1. [Color online.] Illustration of the two most relevant types of solutions we expect to find in the networks here considered. A synchronous solution whose trajectory is represented by the black dashed line, which lies on the synchronization manifold, and the desynchronous solution whose trajectory is represented by the large gray (red online) filled regions. The sum of the positive Lyapunov exponents of the synchronous solution is denoted by Λ_C and the sum of the positive Lyapunov exponents of the desynchronous solution is denoted by Λ.

filled gray (red online) regions. After reaching this attractor, we calculate the Lyapunov exponents of a trajectory along this typical attractor. We represent the sum of all the positive Lyapunov exponents by Λ. Such a typical attractor represents two coupled systems that are NOT completely synchronous. Their trajectories are always different, $x_i \neq x_j$. Even when such a typical attractor asymptotically stable exists, the trajectory that remains along the synchronization manifold is a coexisting solution of the two coupled systems. There are two solutions. One typical desynchronous ($x_i \neq y_i$, for all i) and asymptotically stable which produces Λ and a coexisting synchronous solution ($x_i = y_i$, for all i) and unstable which produces Λ_C. The point we want to make is that as long as a synchronization manifold exists, we have at least two coexisting solutions.

If the synchronization manifold is stable, initial conditions close to the synchronization manifold eventually falls on the synchronization manifold and remains there forever. Trajectories for which $x_i \neq y_i$ for all i might not exist any longer. The only attracting set might be the synchronization manifold. In that case, complete synchronization is achieved, the Lyapunov exponents of typical initial conditions are equal to the conditional Lyapunov exponents, and therefore $\Lambda = \Lambda_C$.

Roughly speaking, our conjecture states that if for two ($N = 2$) coupled nodes with equal dynamics and given coupling strengths, the quantity Λ is greater (smaller) than Λ_C, then this inequality remains valid for $N > 2$ coupled nodes with equal dynamics for coupling strengths obtained by a proper rescaling.

3. Dynamical networks

Consider a dynamical network formed by $N > 0$ equal nodes $\mathbf{x}_i \in \mathbb{R}^d$ with $d > 2$. The network is described by

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} \mathcal{G}_{ij}\mathbf{H}(\mathbf{x}_j) - g\sum_{j=1}^{N} \mathcal{C}_{ij}\mathbf{S}(\mathbf{x}_i, \mathbf{x}_j), \tag{1}$$

where $g \in \mathbb{R}$ and $\sigma > 0$ are the linear and non-linear coupling strengths among the nodes, respectively. $\mathcal{G} = \{\mathcal{G}_{ij}\}$ is a Laplacian matrix ($\sum_j \mathcal{G}_{ij} = 0$) describing the way nodes are linearly coupled, $\mathcal{C} = \{\mathcal{C}_{ij}\}$ is the adjacent matrix representing the way the nodes are connected by linear and non-linear functions, and $\mathbf{H}: \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{S}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are arbitrary differentiable transformations. We also assume that $\mathcal{G}$ and $\mathcal{C}$ commute.

A solution of (1) is called synchronous if $\mathbf{x}_1(t) = \cdots = \mathbf{x}_N(t)$. To guarantee the existence of such solutions, we assume that every node of the network receives the same number k of incoming connections. In other words, we require that $\sum_j \mathcal{C}_{ij} = k$ for any i. It is easy to see that this condition not only guarantees the existence of synchronous solution, but also implies that the d-dimensional linear subspace $\mathcal{S} = \{\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\}$ is invariant. The set $\mathcal{S}$ is called synchronization manifold. Note that a synchronous solution $\mathbf{x}_i(t) = \mathbf{x}(t)$ for $i = 1, \ldots, N$ satisfies the following ordinary differential equation

$$\dot{\mathbf{x}} = F(\mathbf{x}) - gk\mathbf{S}(\mathbf{x}, \mathbf{x}). \tag{2}$$

4. Calculation of Lyapunov exponents and of conditional Lyapunov exponents

The way small perturbations $\delta\mathbf{x}_1, \delta\mathbf{x}_2, \ldots, \delta\mathbf{x}_N$ propagate in the network is described by the variational equations [7] associated to (1)

$$\dot{\delta\mathbf{x}}_i = DF(\mathbf{x}_i)\delta\mathbf{x}_i + \sigma\sum_{j=1}^{N} \mathcal{G}_{ij} D\mathbf{H}(\mathbf{x}_j)\delta\mathbf{x}_j + g\sum_{j=1}^{N} \mathcal{C}_{ij} D_1\mathbf{S}(\mathbf{x}_i, \mathbf{x}_j)\delta\mathbf{x}_i - g\sum_{j\neq i} \mathcal{C}_{ij} D_2\mathbf{S}(\mathbf{x}_i, \mathbf{x}_j)\delta\mathbf{x}_j, \tag{3}$$

where $D_1S(x, y)$ and $D_2S(x, y)$ denote the differential of $S(x, y)$ with respect to x and y, respectively. From (3), we can calculate the Lyapunov exponents of every solution of (1).

4.1. The Lyapunov exponents and the quantity Λ

To calculate the Lyapunov exponents (LEs), we set the initial conditions of the nodes to be non-equal and then we calculate the Lyapunov exponents using the variational equation (3). We assume that the considered random initial conditions evolve always to the same asymptotic attractor. In a more precise way, we assume that the network is ergodic, and so the Lyapunov exponents $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{Nd}$ are constant almost everywhere, and can be obtained by typical initial conditions. Therefore, Lyapunov exponents measure how much nearby trajectories exponentially diverge in a typical asymptotic attractor.

Then,

$$\Lambda = \sum_m \lambda_m^+ \tag{4}$$

where λ_m^+ represents the m positive LEs.

4.2. The conditional Lyapunov exponents and the quantity Λ_C

The Lyapunov exponents of the solutions that lie on the synchronization manifold are called conditional Lyapunov exponents (CLEs) and they are represented by $\lambda^{(1)} \geqslant \lambda^{(2)} \geqslant \cdots \geqslant \lambda^{(Nd)}$. To calculate the conditional Lyapunov exponents, we set equal initial conditions for all the nodes of the network, and use Eq. (3) to calculate them. Alternatively, we can also calculate the CLEs by diagonalizing the variational equation in Eq. (3) making it a set of equations for the eigenmodes as in Eq. (A.3) (see also Ref. [18]). The variational equations in the eigenmode form preserve the form even when one considers large networks. And one can use the conditional exponents calculated for two bidirectionally coupled nodes in order to calculate the conditional exponents of larger networks. Usually, the CLEs for two coupled systems must be calculated numerically. Then, the numerically calculated CLEs can be used to calculate all the CLEs of larger dynamical networks.

The chaotic set lying on the synchronization is not a typical set when this manifold is unstable, in the sense that initial conditional arbitrarily close to the synchronization manifold diverge from it. If the synchronization manifold is stable, initial conditions in the neighborhood of the manifold are attracted to it.

We assume that the equal initial conditions lying along the synchronization manifold we set for all the nodes is a typical initial condition. By typical we mean that it provides a set that is unique. And we expect that other initial conditions taken by chance will also lead the trajectory to this unique set. A non-typical initial condition is for example $(x_i, y_i) = (0, 0)$. In a more rigorous sense, we assume that the dynamics restricted to the synchronization manifold $\mathcal{S}$ is ergodic. Hence, also the conditional Lyapunov exponents along synchronous solutions are constant almost everywhere on $\mathcal{S}$ along this manifold. The ergodic invariant measure of (1) and that of the dynamics restricted to $\mathcal{S}$ (not necessarily the same) are assumed to be unique (singular) and different than a point (non-atomic).

Then,

$$\Lambda_C = \sum_m \lambda_+^{(m)} \tag{5}$$

where $\lambda_+^{(m)}$ represent the m positive CLEs.

5. Conjecture

Here, we describe our proposed conjecture in a more friendly way. For a more rigorous presentation of it, one should read Appendix A.1.

Let $\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g, N$ as in (1) to be the parameters which define the dynamical network. $\mathbf{H}$ represents the function under which the nodes connect among themselves in a linear fashion, $\mathbf{S}$ the function under which the nodes connect among themselves in a non-linear fashion, $\mathcal{G}$ a Laplacian connecting matrix, $\mathcal{C}$ an adjacent connecting matrix, σ the strength of the linear coupling and g the strength of the non-linear coupling. Finally, N is the number of nodes.

We say that a network is of the class

- UPPER, if $\Lambda_C \geqslant \Lambda$;
- LOWER, if $\Lambda_C \leqslant \Lambda$.

We consider that the UPPER and LOWER property holds for a properly rescaled coupling strength intervals $\sigma(N, \mathcal{G}, \mathcal{C}) \in [\sigma_m(N, \mathcal{G}, \mathcal{C}), \sigma^*(N, \mathcal{G}, \mathcal{C})]$ and $g(N, \mathcal{G}, \mathcal{C}) \in [g_m(N, \mathcal{G}, \mathcal{C}), g^*(N, \mathcal{G}, \mathcal{C})]$.

Conjecture. *The LOWER or UPPER character of a dynamical network as the one described by Eq. (1) is independent of the number of nodes for a properly rescaled coupling strength interval.*

In simple words, this conjecture states that as long as one preserves the coupling functions $\mathbf{H}, \mathbf{S}$ under which nodes connect among themselves, there will be coupling strengths σ, g for which the LOWER or UPPER character of a dynamical network will be preserved, regardless of the number of nodes N.

6. Defining the coupling strength intervals

For simplicity in the notation, we omit in the representation of the constants σ_m, σ^* and g_m, g^* the reference to their dependence on $\mathcal{G}, \mathcal{C}$.

Our conjecture states that whenever there is a network with N_1 nodes with a structure defined by $\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}$ and this network has an UPPER (or LOWER) character for the coupling strength intervals $[\sigma_m(N_1), \sigma^*(N_1)]$ and $[g_m(N_1), g^*(N_1)]$ then if a network with N_2 nodes is constructed preserving the coupling functions $\mathbf{H}, \mathbf{S}$ then there exists coupling strength intervals $[\sigma_m(N_2), \sigma^*(N_2)]$ and $[g_m(N_2), g^*(N_2)]$ for which the network behaves with the same UPPER (or LOWER) character.

To make this conjecture more practical, we make in the following some assumptions.

The values of the constants $\sigma_m(N), \sigma^*(N)$ and $g_m(N), g^*(N)$ are such that either $|\sigma_m(N)/g_m(N)| \gg 1$ or $|\sigma_m(N)/g_m(N)| \ll 1$ and $|\sigma^*(N)/g^*(N)| \gg 1$ or $|\sigma^*(N)/g^*(N)| \ll 1$. The reason is because for such conditions, the values for these constants for a network with $N_2 > 2$ nodes can be easily calculated from the values of these constants from the reference network.

The network with N_1 nodes is regarded to be the *reference* network and we consider that $N_2 > N_1$. For simplicity, we further consider that $N_1 = 2$. In addition, to make our analysis simpler, we consider in our numerical simulations a constant $g_m(N) = g^*(N)$, and we choose either $|\sigma_m(N)/g_m(N)| \gg 1$ or $|\sigma_m(N)/g_m(N)| \ll 1$. So, we make the non-linear coupling strength constant.

Then, we choose the constant $\sigma^*(N)$ such that its value is a little bigger than the smallest coupling values for which complete synchronization is reached and when $\Lambda = \Lambda_C$. However, other intervals could be considered. The reason again is that the linear coupling strength, $\sigma^*(N)$, for which complete synchronization appears in a network with N nodes can be analytical calculated from the linear coupling strength, $\sigma^*(N = 2)$, for which complete synchronization appears in a network with 2 nodes by using

$$\sigma(N) = \frac{2\sigma(N = 2)}{|\gamma_2(N)|}, \tag{6}$$

$$g(N) = \frac{g(N = 2)}{k} \tag{7}$$

where γ_2 is the second largest eigenvalue of $\mathcal{G}$, and k is the number of incoming connections of each node of the network.

Let us give an example of how we use Eq. (6). Having defined that two mutually linearly coupled systems (so, $g = 0$) have a LOWER character for the linear coupling strength interval $[\sigma_m(N = 2), \sigma^*(N = 2)]$, then we construct a network using the same linear coupling function composed of N nodes, but considering now the linear coupling strength interval $[\sigma_m(N), \sigma^*(N)]$ calculated using Eq. (6). According to our conjecture, such a network will have a LOWER character.

For a more detailed analysis of how we derive Eqs. (6) and (7), one should read Appendix A.2. In Ref. [18], we show that these equations are valid even when $|\sigma_m(N)/g_m(N)| \approx 1$ and $|\sigma^*(N)/g^*(N)| \approx 1$.

7. Networks of coupled maps

Here, we consider only linear couplings. Then $g = g_m = 0$. We also consider that the minimal linear coupling strength is $\sigma_m = 0$.

For general networks (discrete or continuous descriptions) whose nodes are completely synchronous, one always have that $\Lambda = \Lambda_C$, a non-generic case for which our conjecture can be proved.

For networks of coupled maps, there is another trivial example when $\Lambda = \Lambda_C$. That happens for networks whose Jacobian is constant as networks formed by linear maps of the type $x_{n+1}^{(i)} = \alpha x_n^{(i)} + 2\sigma \sum_{j=1}^{N} \mathcal{G}_{ij} x_n^{(j)} \pmod 1$ and when there exists complete synchronization, and the attractor lays on the synchronization manifold. These results concern arbitrary connecting Laplacian matrices $\mathcal{G}_{ij}$, for example, they would apply for map lattice with a

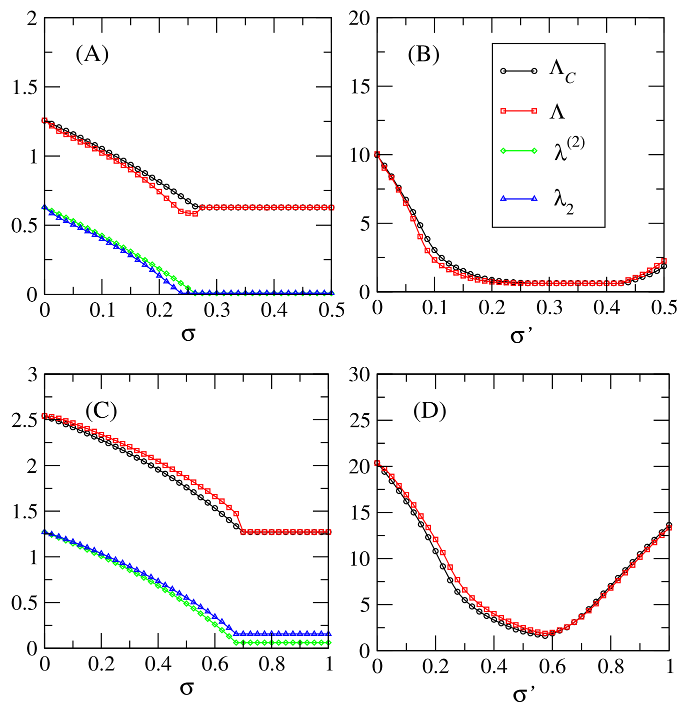

Fig. 2. Results for the network in Eq. (8), for $\rho = 0.5$. For (A) and (C), $N = 2$, and for (B) and (D), $N = 16$. An inhibitory (UPPER) network is shown in (A) and (B), for $s = -1$, and an excitable (LOWER) network is shown in (C) and (D), for $s = 1$. The horizontal axis in (B) and (D) were rescaled by $\sigma' = \sigma * |\gamma_2(N = 16)|/2$, so that one can compare (B) and (D) with (A) and (C). $|\gamma_2(N = 16)| = 4.1542$.

coupling whose strength decreases with the distance as a power-law [19].

Now, imagine the following network

$$x^{(i)}_{n+1} = 2x^{(i)}_n + s\rho {x^{(i)}_n}^2 + 2\sigma \sum_{j=1}^{N} \mathcal{G}_{ij} x^{(j)}_n \ (\mathrm{mod}\, 1) \tag{8}$$

with $\rho \geqslant 0$ and $s = \pm 1$. The synchronization manifold is defined by $x^{(1)}_n = x^{(2)}_n = \cdots = x^{(N)}_n$, and in an all-to-all connecting topology, the Lyapunov exponent of the synchronization manifold can be calculated by $\lambda^{(1)} = \ln(2) + 1/t \sum_n \ln|1 + s\rho x_n|$, with $n = (1, \dots, t)$, and the others $N - 1$ equal exponents associated to the transversal directions by $\lambda^{(i)} = \ln(2) + 1/t \sum_n \ln|1 + s\rho x_n - 2\sigma|$, for $i \geqslant 2$. In Fig. 2, we show the values of Λ and Λ_C as we vary σ, for $\rho = 0.5$. In (A) and (C), we consider $N = 2$ (all-to-all topology), and in (B) and (D) we consider a random networks formed by $N = 16$ nodes. The coupling strength interval used for two coupled nodes was rescaled to the proper coupling strength interval for the larger random network, using in the denominator of Eq. (6) the value of $|\gamma_2| = 4.1542$, relative to the second largest eigenvalue (in absolute value) of the random network. One can check that if two coupled nodes have an UPPER [LOWER] character for a given coupling interval as can be seen in Fig. 2(A) [in Fig. 2(C)], larger networks will behave in the same UPPER [LOWER] character as can be seen in Fig. 2(B) [in Fig. 2(D)].

The conjecture describes a relationship between the conditional exponents and the Lyapunov exponents. To see that, notice that, typically for the UPPER networks of linearly connected maps, we have $\lambda_1 \approx \lambda^{(1)}$, a consequence of the fact that the largest Lyapunov exponent can be calculated using the same directions as the ones along the synchronization manifold. Thus, using our conjecture, if the network is of the UPPER type, $\lambda_1 + \lambda_2 \leqslant \lambda^{(1)} + \lambda^{(2)}$, which provides $\lambda_2 \leqslant \lambda^{(2)}$. Otherwise, if the network is of the LOWER type, $\lambda_2 \geqslant \lambda^{(2)}$. That can be checked in Figs. 2(A)–(C). Since the approaching of the transversal conditional exponents to negative values are associated with the stabilization of a certain oscillation mode, close to a coupling strength for which a transversal conditional exponent approaches zero, there will also be a Lyapunov exponent which approaches zero, meaning that some oscillation in the attractor becomes stable.

8. Networks of Hindmarsh–Rose neurons

Let us illustrate our conjecture in networks composed of N coupled Hindmarsh–Rose neurons [17] electrically and chemically coupled[5]:

$$\dot{x}_i = y_i + 3x_i^2 - x_i^3 - z_i + I_i - g \sum_{j=1}^{N} \mathcal{C}_{ij} S(x_i, x_j) + \sigma \sum_{j=1}^{N} \mathcal{G}_{ij} x_j,$$
$$\dot{y}_i = 1 - 5x_i^2 - y_i, \qquad \dot{z}_i = -rz_i + 4r(x_i + 1.6). \tag{9}$$

The parameter r modulates the slow dynamics and is set equal to 0.005, such that each neuron is chaotic. The synaptic chemical coupling is modeled by $S(x_i, x_j) = (x_i - V_{syn})\Gamma(x_j)$ where $\Gamma(x_j) = \frac{1}{1+e^{-\theta(x_j - \Theta_{syn})}}$ with $\Theta_{syn} = -0.25$, $\theta = 10$ and $V_{syn} = 2.0$. $\sigma \mathcal{G}_{ji}$ is the strength of the electrical coupling between the neurons, and $I_i = 3.25$. In order to simulate the neuron network and to calculate the Lyapunov exponents through Eq. (A.2), we use for the node i the initial conditions $x_i = -1.3078 + \omega_i$, $y_i = -7.3218 + \omega_i$, and $z_i = 3.3530 + \omega_i$, where ω_i is a uniform random number within [0,0.02]. To calculate the conditional exponents $\lambda^{(i)}$, we use in Eq. (A.3) the initial conditions, $x = -1.3078$, $y = -7.3218$, and $z = 3.3530$, but other set of typical equal initial conditions can be used.[6]

We study three types of neural networks:

Case (i): $g < 0$ [Figs. 3(A)–(C)]. The coupling (synapses) is said to be of the inhibitory type, since $(x_i - V_{syn}) < 0$ and the nodes j contribute negatively in the equations for the first derivative of x_i. In other words, the post-synaptic neuron (x_i) is forced to synchronize its rhythm to the rhythm of the pre-synaptic ones (x_j).

Case (ii): $g = 0$ [Figs. 3(D)–(F)]. The network has nodes coupled to other nodes only electrically. From the biological point of view, neurons only make electrical connections with their nearest neighbors. Here, we also consider that neurons can make long-range electrical couplings, and so neurons can connect electrically with other non-neighbor nodes. Since $\sigma \geqslant 0$, this coupling contributes effectively negatively to the first derivative of x_i, which results in an inhibitory effect to the oscillatory motion of the neuron x_i.

Case (iii): $g > 0$ [Figs. 3(G)–(I)]. The coupling (synapses) is said to be of the excitatory type, since the nodes j contribute posi-

[5] It is interesting to observe here that this widely employed synaptic chemical coupling function can be written as $\Gamma(x_j) = 1 - F(x_j)$, with $F(x_j) = 1/(1 + \exp[\theta(x_j - \Theta_{syn})])$. In this way, one may interpret the term $F(x_j)$ as a Fermi distribution with $1/\theta$ acting as a temperature and Θ as the chemical potential. Such a distribution is a commonality in quantum statistics of Fermion particles obeying the exclusion principal: no more than one particle (here a neuron) can occupy the same state.

[6] Our conjecture applies to networks for which for every chaotic attractor that possesses a basin of attraction with a positive measure there exists also an unstable chaotic saddle (the solution lying on the synchronization manifold) associated to this chaotic attractor. In the case there exists multiple chaotic attractors, the coupling strengths $\sigma^*(N)$ and $g^*(N)$ (as well as $\sigma_m(N)$ and $g_m(N)$) would be a function of the chosen basin of attraction. The requirement we make is that any initial condition belonging to an open neighborhood around the unstable chaotic saddle goes to only one chaotic attractor. These initial conditions are regarded as the typical ones. Therefore, in our simulations we consider initial conditions that are small perturbations around the synchronization manifold. Nevertheless, most of the attractors obtained are completely out of synchrony. The identification of possible many coexisting attractors is a technical matter. In a very general situation, there will be hopefully only a few coexisting attractors and one can clearly identify the functions $\sigma^*(N)$ and $g^*(N)$ (as well as $\sigma_m(N)$ and $g_m(N)$).

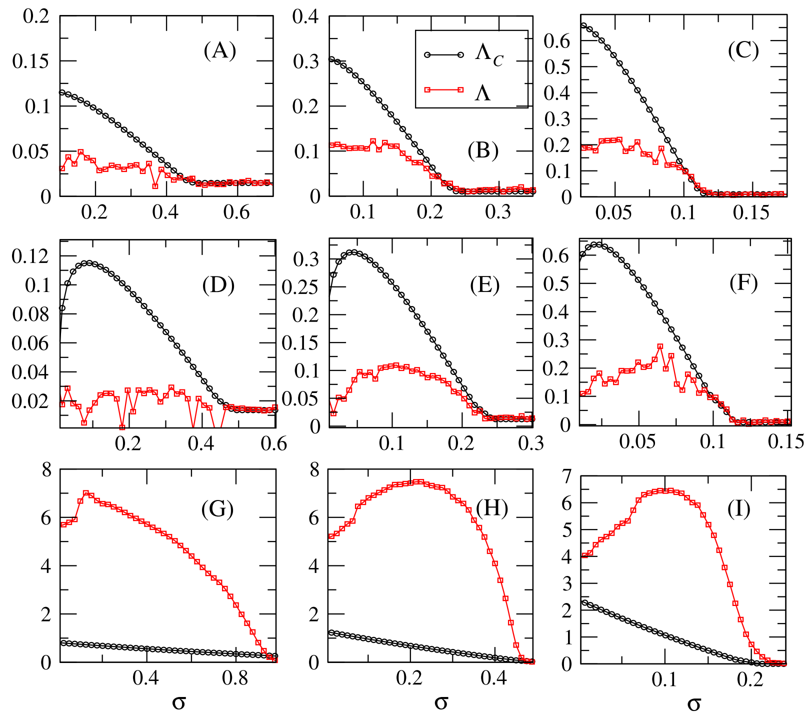

Fig. 3. The values of Λ and Λ_C for neural networks described by Eq. (9) of nodes connected in an all-to-all topology. In (A), (D), and (G), $N = 2$. In (B), (E), and (H), $N = 4$. In (C), (F), (I), $N = 8$. Results for networks with an UPPER character are shown in (A)–(F), and for networks with a LOWER character are shown in (G)–(I).

tively in the equations for the first derivative of x_i. For such a case, the post-synaptic neuron (x_i) is forced to opposite the pre-synaptic ones (x_j). The way we define excitability in this work is different from the realistic biological way, where excitability is defined in terms of the parameter $V_{syn} = -2.0$. But the results presented here are similar to the results obtained for excitatory networks whose neurons have a biologically plausible excitatory synapse. One can check that in Ref. [18].

In Fig. 3, we show the values of Λ and Λ_C for the three types of neural networks being considered, Case (i) in Figs. 3(A)–(C), Case (ii) in Figs. 3(D)–(F), and Case (iii) in Figs. 3(G)–(I). Networks whose results are represented in Figs. 3(A)–(C) and (G)–(I) are constructed by neurons connected both electrically ($\sigma > 0$) and chemically in the all-to-all topology, while networks whose results are represented in Figs. 3(D)–(F) are constructed by neurons connected only electrically ($\sigma > 0$ and $g = 0$) in the all-to-all topology.

In (A) [Case (i)], for $N = 2$ and $g = -0.01$, $\Lambda \leqslant \Lambda_C$, for $\sigma = [0.1, 0.7]$. So, $\sigma_m(N = 2) = 0.1$ which leads to $|\sigma_m(N = 2)/g_m(N = 2)| \gg 1$, as we wish. From our conjecture, for larger networks as the ones shown in Figs. 3(B) [$N = 4$] and 3(C) [$N = 8$], we must have $\Lambda \leqslant \Lambda_C$, for the rescaled coupling interval. From Eqs. (6) and (7), we have that for the network with $N = 4$ [Fig. 3(B)], the rescaled coupling strength interval should be $\sigma = [0.1/2, 0.7/2]$ and $g = -0.01/3$, and for the network with $N = 8$ [Fig. 3(C)], the rescaled coupling strength interval should be $\sigma = [0.1/4, 0.7/4]$ and $g = -0.01/7$. In fact, as one sees in Figs. 3(B)–(C), these networks have the same UPPER character as the network with $N = 2$, for the considered coupling strength intervals.

In (D) [Case (ii)], for $N = 2$ and $g = 0$, $\Lambda \leqslant \Lambda_C$ for $\sigma = [0, 0.6]$. So, $g_m(N = 2) = 0$ and consequently $\sigma_m(N = 2) = 0$. From our conjecture, for larger networks as the ones shown in Figs. 3(E) [$N = 4$] and 3(F) [$N = 8$], we must have $\Lambda \leqslant \Lambda_C$ for the rescaled coupling interval. From Eqs. (6) and (7), we have that for $N = 4$ [Fig. 3(E)], the rescaled coupling interval should be $\sigma = [0, 0.6/2]$ and for $N = 8$ [Fig. 3(F)], the rescaled coupling interval should be $\sigma = [0, 0.6/4]$. In fact, as one sees in Figs. 3(E)–(F), these networks have the same UPPER character of the network with $N = 2$.

Finally, In (G) [Case (iii)], for $N = 2$ and $g = 10$, $\Lambda \geqslant \Lambda_C$ for $\sigma = [0.01, 1]$. So, $|g_m(N = 2)/\sigma_m(N = 2)| \gg 1$ as we wish. From our conjecture, for larger networks, as the ones shown in Figs. 3(H) [$N = 4$] and 3(I) [$N = 8$], we must have $\Lambda \geqslant \Lambda_C$ for the rescaled coupling interval. From Eqs. (6) and (7), and $N = 4$ [Fig. 3(H)], the rescaled coupling interval should be $\sigma = [0.01/2, 1/2]$ and $g = 10/3$, and for $N = 8$ [Fig. 3(I)], the rescaled coupling interval should be $\sigma = [0.01/4, 1/4]$ and $g = 10/7$. In fact, as one see in Figs. 3(G)–(I), these networks have the same LOWER character of the network with $N = 2$.

An inhibitory chemical coupling inhibits the nodes of the network, which means that such a coupling promotes an increase in the level of synchronization. On the other hand, an excitatory chemical coupling excites the nodes, which means that such a coupling promotes an increase in the level of desynchrony. We have previously shown in Figs. 3(A)–(C) that an inhibitory network (as usually defined in terms of the chemical coupling) has the UPPER characteristic and in Figs. 3(G)–(I) that an excitatory network (as usually defined in terms of the chemical coupling) has the LOWER characteristic.

Had we considered that the neurons were connected exclusively by non-linear (chemical) means ($\sigma = 0$), then it is to be expected that inhibitory networks would present an UPPER character and excitatory networks would present a LOWER character.

9. Application of our conjecture to predict the chaotic behavior of large networks

In the following, we discuss how our conjecture can be used to make general statements about dynamical networks.

9.1. Calculating an upper bound for the Kolmogorov–Sinai entropy

In this subsection we will show that a dynamical network whose N nodes are linearly connected in an all-to-all fashion produces more information (Kolmogorov–Sinai entropy) than similar networks (same **H**, **S**, **N**), but with any other possible connecting topologies.

Consider the UPPER networks ($\Lambda_C \geqslant \Lambda$) formed by neurons connected only electrically ($g = 0$). For such cases, $\Lambda_C(N)$ is an upper bound for the Kolmogorov–Sinai entropy, denoted by H_{KS} (see footnote 3), since $H_{KS} \leqslant \Lambda$.

Networks formed by nodes connected in an all-to-all topology produce Laplacian matrices whose eigenvalues are $\gamma_1 = 0$, and $\gamma_i = -N$, for $i = 2, \ldots, N$. From Eq. (A.4) one can conclude that $\max[\Lambda_C(N)]$ for the considered coupling strengths of a network with the all-to-all topology, is larger or equal to $\max[\Lambda_C(N)]$ for any other connecting topology.

To understand the reason we need to turn into Eq. (A.4) that allow us to calculate the CLEs of larger networks from the CLEs of two coupled nodes. Assume that the maximal value of $\lambda^{(2)}$ for two coupled neurons happens when the coupling strength is $\tilde{\sigma}(N = 2)$. Thus, from Eq. (A.4), the maximal value of the i-th CLE, $\lambda^{(i)}$, happens for when

$$\tilde{\sigma}(N) = \frac{2\tilde{\sigma}(N=2)}{|\gamma_i(N)|}.$$

But, since that for the all-to-all topology, $\gamma_i = -N$, then all CLEs $\lambda^{(i)}$ for $i \geqslant 2$ will happen for the same coupling strength $\tilde{\sigma}(N)$. The quantity Λ_C is calculated for a particular coupling strength. When we calculate Λ_C for $\tilde{\sigma}(N)$, we obtain that Λ_C is a sum of maximal values for $\lambda^{(i)}$. As a consequence, Λ_C is the maximal possible value one can ever have, considering all other possible topologies. For other topologies, γ_i typically differ and as a consequence for a given coupling strength some $\lambda^{(i)}$ might be maximal, others might be smaller than this maximal value, leading to a smaller Λ_C.

So, we can define the *network capacity*, $c(N)$, as

$$c(N) = \max\left[\Lambda_C(N)\right] \tag{10}$$

calculated for the all-to-all topology (and the considered coupling intervals).

Since $\Lambda_C(N) \geqslant \Lambda(N)$ (as well as $\Lambda_C(L) \geqslant H_{KS}(N)$ [20]) for UPPER networks, we conclude that for these networks not only

$$c(N) \geqslant \max\left[\Lambda(N)\right] \tag{11}$$

but also

$$c(N) \geqslant \max\left[H_{KS}(N)\right] \tag{12}$$

where the max of $\Lambda(N)$ in taken considering "any" possible topologies (described in Fig. 4) and the considered coupling intervals.

The value of $c(N)$ for neural networks electrically connected can be calculated by $\max(\lambda^{(1)}) + (N-1)\max(\lambda^2)$. Notice that since $\lambda^{(1)}$ is a constant value for all σ (it does not depend on it), then $\max(\lambda^{(1)})$ happens for the same coupling strength for which $\max(\lambda^{(2)})$ is found, which leads to

$$c(N) \cong 0.01362 + 0.1013(N-1)\ \text{bits/(time unit)}. \tag{13}$$

To numerically verify Eq. (11), we do simulations considering networks as the ones represented in Fig. 4, (with $10 \leqslant N \leqslant 40$), excluding the nearest neighbor connecting topology (Fig. 4(A)) and the all-to-all topology (Fig. 4(D)). We obtain that $\max[\Lambda(N)] \cong 0.0830 + 0.0230(N-1)$ bits/(time unit). Therefore, as expected $c(N) > \max[\Lambda(N)] > H_{KS}$.

For a network with the all-to-all topology [as in Fig. 4(D)], for $10 \leqslant N \leqslant 40$, we obtain $\max[\Lambda(N)] \cong 0.158447 + 0.031537(N-1)$, which agrees with Eq. (11), because $c(N) \geqslant \max[\Lambda(N)]$, where the maximum is taken considering the all-to-all topology and varying the coupling strengths.

Finally, if we construct a network with nodes connecting to their nearest neighbors forming a closed ring [as in Fig. 4(A)], we find $\max[\Lambda(N)] \cong 0.197125 + 0.034865(N-1)$ bits/(time unit). Eq. (11) is once again verified.

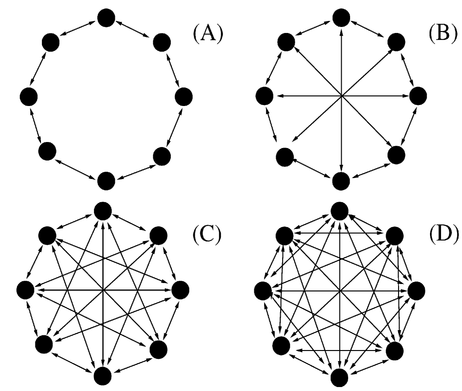

Fig. 4. Representation of a few network topologies with 8 neurons, considered in this work. The filled balls represent neurons and the lines indicate an electric bidirectional coupling. In (A) the neurons are only coupled with its nearest neighbors, forming a ring. From (B) to (D) it is added to the network long-range bidirectional connections, The average number of connections that each neuron receives (network degree), ω, is $\omega = 2$, in (A), $\omega = 3$, in (B), $\omega = 5$, in (C), and $\omega = 7$, in (D). In a network with N neurons, long-range connections are introduced in the initial ring by connecting each neuron to its $N/2$-th (B) neighbors, then to its $(N/2-1)$-th neighbors (C), then to its $(N/2-l)$-th neighbors, till each neuron is connected to its second neighbors, when the network has the all-to-all coupling topology.

Thus, $c(N)$ for linearly (electrically) connected networks, does not depend on the network topology. That is not the case for chemically connected neural networks, for which $c(N)$ might be achieved for different topologies, since the curves for $\lambda^{(1)}$ and $\lambda^{(i)}$ with respect to the coupling strength σ achieve their maximal values for different values of σ.

9.2. An upper bound for the Kolmogorov–Sinai entropy in networks with non-equal nodes

It is interesting to investigate whether in an UPPER network the quantity Λ_C remains as an upper bound for the Kolmogorov–Sinai entropy of a dynamical network with nodes that have parameter mismatches, a situation to be expected in physical networks.

In order to understand that, we consider networks composed by $N =$[10,20,30] neurons connected in an all-to-all topology, and set the parameter $I_i = 3.2 + \eta\xi$, where ξ is a random number uniformly distributed between 0 and 1 and η represents the amplitude with which the parameter I_i is randomly varied. Then, we vary the linear coupling strengths within an interval were $\max[\Lambda(N)]$ is found. This interval is defined by $\sigma = [0.001 * 2/N, 0.4 * 2/N]$, being that complete synchronization between N mutually coupled neurons is achieved for $\sigma \geqslant 2 * 0.4/N$.

In Fig. 5, we show the value of Λ as a function of the rescaled coupling strength $\frac{N\sigma}{2}$. From (A) to (C) we show results concerning networks with $N = 10$, $N = 20$, and $N = 30$ nodes, respectively. The different curves represent results from different noise amplitudes η considered when generating the parameter values of each neuron in the network. Filled circles represent $\eta = 0$, empty squares $\eta = 0.2$, stars $\eta = 0.4$, and pluses $\eta = 0.6$.

From this figure, we notice that as the neurons are set with larger parameter mismatches (larger η), for a large interval of the coupling amplitude σ, the value of Λ as well as the maximal value of it decrease. In larger networks (larger N), for a large interval of the coupling strength σ, the difference between $\Lambda(N, \eta = 0)$ and $\Lambda(N, \eta > 0)$ is larger than this difference in smaller networks. This effect is a consequence of the fact that as the number of nodes increases, the effect of the different nodes on the value of Λ is amplified. Since for UPPER networks we have that $\Lambda_C(N, \eta = 0) \gg \Lambda(N, \eta = 0)$ and as shown numerically we have

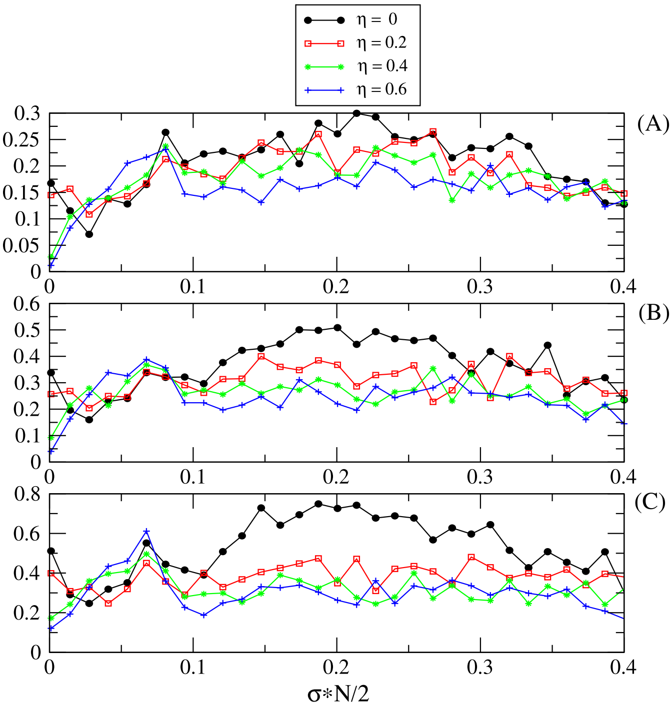

Fig. 5. We show the value of Λ as a function of the rescaled coupling strength $\frac{N\sigma}{2}$. From (A) to (C) we show results concerning networks with $N = [10,20,30]$. Filled circles represent $\eta = 0$, empty squares $\eta = 0.2$, stars $\eta = 0.4$, and pluses $\eta = 0.6$.

that for most of the coupling strengths $\Lambda(N, \eta = 0) > \Lambda(N, \eta > 0)$ and for some other parameters $\Lambda(N, \eta = 0) \cong \Lambda(N, \eta > 0)$, we conclude that $\Lambda_C(N, \eta = 0) \geqslant \Lambda(N, \eta > 0)$.

Therefore the maximal rate of information produced by a complex network constructed with equal nodes ($\Lambda_C(N, \eta = 0)$) is larger or equal than the rate of information of an equivalent dynamical network (same topology and same number of nodes) but whose nodes have parameter mismatches $\Lambda(N, \eta > 0)$.

These results were similarly reproduced in some other networks with the connecting topologies represented in Fig. 4.

9.3. Predicting whether a large dynamical network will be chaotic

Further, consider two coupled LOWER-type systems and Λ is null (positive) for some coupling strength, meaning a periodic behavior (meaning chaos). It might be that, for a proper rescaled coupling strength, as more nodes are added to the network, Λ becomes positive, meaning chaos (for sure there will be chaos). We can also use our conjecture to predict the behavior of a network constructed with nodes that are either chaotic or periodic, by only having information about two coupled nodes. Considering only linear couplings [$g = 0$, in Eq. (1)]. For $\sigma \leqslant \epsilon$, the two coupled nodes have a periodic dynamics, and thus, $\Lambda = 0$, but $\Lambda_C > 0$ (UPPER character). That implies that as we add more nodes in the network, it might be that after the proper rescaling of the coupling strength the network becomes chaotic.

10. Conclusions

In conclusion, we have presented arguments to suggest that for a class of dynamical systems, the sum of all the positive Lyapunov exponents of a dynamical network is bounded by the sum of all the positive Lyapunov exponents of the synchronization manifold. In practical terms, the entropy production of the synchronization manifold and its transversal directions (Λ_C) of a system of two coupled equal dynamical systems determines the upper (LOWER character) or lower (UPPER character) bound for the sum of the positive Lyapunov exponents of a large network. This fact enables one to predict the behavior of a large network by using information provided by only two coupled nodes.

Our results indicate that the behavior (synchronization and information) of a dynamical network with nodes possessing equal dynamics and especial properties see footnote 6 does not strongly depend on the coupling topology ($\mathcal{G}$ and $\mathcal{C}$) and the size of the network (N) but rather on the nature of the coupling functions (S and H).

At first glance, this result seems to be in direct conflict with what one would expect to find in realistic neural networks, as the mammalian brain, whose topology is possibly responsible for intelligence. But one should have in mind that the here considered networks are constructed with nodes that possess equal dynamics being connected using always the same coupling function. In realistic brain networks, the coupling functions largely differ along different brain areas as well as the coupling strength depends on time. Therefore, in order for the topology to play an important role in the behavior of a network one needs to consider networks with parameter mismatches and/or that possess coupling functions that change in space and time.

Naturally, the large class of networks for which our conjecture was constructed are far from being realistic. However, our conjecture can contribute to the understanding of much more complex dynamical networks. For example, for the UPPER networks, we have numerically shown in Section 9.2 that the maximal rate of information produced by a dynamical network constructed with equal nodes ($\Lambda_C(N, \eta = 0)$) is larger or equal than the rate of information of an equivalent UPPER dynamical network (same topology and same number of nodes) but whose nodes have parameter mismatches $\Lambda(N, \eta > 0)$. This result complements a previous result from Ref. [21] where we have numerically shown that networks whose nodes are connected linearly and have mismatches in the coupling strengths produce less information than networks whose coupling strengths are equal. Therefore, even though networks with equal nodes (connected with equal coupling strengths) might not be realistic, their entropy production is an upper bound for the entropy production of more realistic networks, if the networks are of the UPPER class. And inhibitory Hindmarsh–Rose neuron networks are of the UPPER class.

Excitability and inhibition is a concept usually used to classify the way non-linear (chemical) synapses between two neurons are done. When an inhibitory neuron spikes (the pre-synaptical neuron) a neuron connected to it (the post-synaptical neuron) is prevented to spike. When an excitatory neuron spikes it induces the post-synaptical neuron to spike. We have shown in this work that an excitatory network is a LOWER network and an inhibitory network is an UPPER network. Thus, in inhibitory networks, the rate of information produced by the network cannot be larger than a value which we can calculate based on the rate of information produced by two mutually coupled neurons. In other words, for UPPER networks, the entropy of the attractors cannot be larger than the entropy of the synchronous set, which therefore imposes a clear limit in the complex character of these networks. On the hand, for LOWER networks, our conjecture suggests that such a limit might be unknown.

As other examples of applications of our conjecture, we show in Section 9.1 how one can calculate an upper bound for the Kolmogorov–Sinai entropy of an UPPER network with equal nodes and whose nodes are fully linearly connected. Then, we show analytically (with numerical verification) that the maximal value of this upper bound (varying the linear coupling strength) is larger than the maximal value of the Kolmogorov–Sinai entropy of an equivalent UPPER network whose nodes are linearly connected with any other connecting topology. And in Section 9.3 we discuss how our conjecture can be used to predict whether a LOWER network formed by nodes that when isolated are chaotic (periodic) will maintain such a chaotic behavior, then predicting how complex larger dynamical networks can be.

This conjecture might be a consequence of the fact that the attractors and behaviors that appear in two coupled nodes for a given coupling strength are similar to the ones that appear for larger networks, to parameters rescaled according to Eqs. (6) and (7). In fact, as one can see in the work [22], that is indeed the case for the coupling strengths for which burst phase synchronization (BPS) or phase synchronization (PS) appear in networks of electrically coupled HR-neurons.

Acknowledgements

M.S.B. acknowledges the partial financial support of "Fundação para a Ciência e Tecnologia (FCT), Portugal" through the programs POCTI and POSI, with Portuguese and European Community structural funds and financial support from the Northern Research Partnership. This work is also supported in part by the CNPq and FAPESP (M.S.H.). M.S.H. is the Martin Gutzwiller Fellow 2007/2008.

Appendix A

A.1. The conjecture

Let $\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g, N$ as in (1) to be the parameters which defines the dynamical network. $\mathbf{H}$ represents the function under which the nodes connects among themselves in a linear fashion, $\mathbf{S}$ the function under which the nodes connects among themselves in a non-linear fashion, $\mathcal{G}$ a Laplacian connecting matrix, $\mathcal{C}$ an adjacent connecting matrix, σ the strength of the linear coupling and g the strength of the non-linear coupling. Finally, N is the number of nodes.

Denote by $\Lambda(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g, N)$ and $\Lambda_C(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g, N)$ the sum of the positive Lyapunov exponents and the sum of the positive conditional Lyapunov exponents of the network whose structure is specified by $(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C})$, respectively. We say that the couple $(\mathbf{H}, \mathbf{S})$ makes the network to be of the LOWER class if for every $(\mathcal{G}, \mathcal{C})$ there exist four positive constants σ_m, g_m, σ^* and g^* such that

$$\Lambda_C(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g) \geqslant \Lambda(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C}, \sigma, g) \tag{A.1}$$

for all $\sigma_m \leqslant \sigma \leqslant \sigma^*$ and all $g_m \leqslant g \leqslant g^*$. An UPPER class dynamical network is defined similarly by reversing the direction of inequality (A.1).

Conjecture. *Given a network with a LOWER (UPPER) character [as defined in (A.1)] specified by $(\mathbf{H}, \mathbf{S}, \mathcal{G}, \mathcal{C})$, and $(\mathcal{G}, \mathcal{C})$ with N_1 nodes, there exist coupling strength intervals $\tilde{\sigma}_m \leqslant \sigma \leqslant \tilde{\sigma}^*$ and $\tilde{g}_m \leqslant g \leqslant \tilde{g}^*$ for which a network specified by $(\mathbf{H}, \mathbf{S}, \tilde{\mathcal{G}}, \tilde{\mathcal{C}})$ and $(\tilde{\mathcal{G}}, \tilde{\mathcal{C}})$ with N_2 nodes has also a LOWER (UPPER) character.*

A.2. Derivation of the coupling strength constants

The variational equation (3) for the synchronous solution can be written as follows

$$\delta\dot{\mathbf{X}} = \big\{\mathbf{I} \otimes D\mathbf{F}(\mathbf{x}) + \sigma\mathcal{G} \otimes D\mathbf{H}(\mathbf{x}) - g\mathcal{C} \otimes D_1\mathbf{S}(\mathbf{x}, \mathbf{x}) - gk\mathcal{C} \otimes D_2\mathbf{S}(\mathbf{x}, \mathbf{x})\big\}\delta\mathbf{X}, \tag{A.2}$$

where $\delta\mathbf{X}$ is the column vector of $\mathbb{R}^{Nd}$ with components $\delta\mathbf{x}_1, \delta\mathbf{x}_2, \ldots, \delta\mathbf{x}_N$, and $\otimes$ stands for the Kronecker product of matrices. Since $\mathcal{G}$ and $\mathcal{C}$ commute, they can be simultaneously diagonalized. Let $\mathbf{u}_1, \ldots, \mathbf{u}_N$ be their eigenvectors, and denote by $\gamma_1, \ldots, \gamma_N$ and $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_N$ the corresponding eigenvalues for $\mathcal{G}$ and $\mathcal{C}$, respectively. We order $\{\gamma_i\}$ so that $\gamma_1 = 0$. If we write $\delta\mathbf{X}(t) = \sum_{1\leqslant i\leqslant N} \mathbf{u}_i \otimes \mathbf{y}_i(t)$ with $\mathbf{y}_i(t) \in \mathbb{R}^d$, and substitute it in (A.2), then a straightforward computation gives

$$\dot{\mathbf{y}}_i = \big\{D\mathbf{F}(\mathbf{x}) + \sigma\gamma_i D\mathbf{H}(\mathbf{x}) - gkD_1\mathbf{S}(\mathbf{x}, \mathbf{x}) - g\tilde{\gamma}_i D_2\mathbf{S}(\mathbf{x}, \mathbf{x})\big\}\mathbf{y}_i. \tag{A.3}$$

While Eq. (A.2) describes how perturbations are propagated or damped along a particular node of the network ($\mathbf{x}_i$) Eq. (A.3) describes how perturbations are propagated along an eigenmode ($\mathbf{y}_i$). While Eq. (A.2) is valid for networks with nodes initially set in typical initial conditions Eq. (A.3) is only valid for networks with nodes initially set with equal initial conditions, the assumption done in order to place Eq. (A.2) in the eigenmode form in Eq. (A.3).

Calculating the Lyapunov exponents from Eq. (3) assuming equal initial conditions for every node provides the same exponents than the conditional ones obtained from Eq. (A.3). An advantage of using Eq. (A.3) for the calculation of the conditional exponents is that while Eq. (A.2) requires the employment of $(Nd \times Nd)$-dimensional matrices, the conditional exponents by Eq. (A.3) requires the use of N matrices of dimensionality d. A mode i in equation in Eq. (A.3) provides a set of d conditional exponents, denoted by $\lambda_j^{(i)}$, $j = 1, \ldots, d$. Since we are only interested in positive exponents, we simplify the notation by making $\lambda^{(i)} = \sum_{j=1}^{d} \lambda_j^{(i)}$. So, $\lambda^{(1)}$ refers to the sum of the positive conditional Lyapunov exponents of the synchronization manifold while $\lambda^{(i)}$ $(i \geqslant 2)$ refer to the sum of the positive Lyapunov exponents of the transversal directions to the synchronization manifold.

From Eq. (A.3) it becomes clear that once the conditional exponents are calculated using two bidirectionally coupled nodes, for the considered coupling interval, the conditional exponents of the mode i ($\lambda^{(i)}$) for larger networks with arbitrary topology can be calculated from the exponents for $N = 2$, by $\lambda^{(1)}(N = 2, \sigma, g) = \lambda^{(1)}(N, \sigma, g/k)$ and $\lambda^{(2)}(N = 2, \sigma, g) = \lambda^{(i)}(N, 2\sigma/|\gamma_i(N)|, g/k)$.

To understand why, just make in Eq. (A.3) $g = 0$. The only term that changes in these equations as one considers networks with different topologies and sizes is $\gamma_i(N)$, the i-th eigenvalue of the connecting Laplacian matrix $\mathcal{G}$ with size N. Denoting $\gamma_i(N = 2)$ and $\sigma(N = 2)$ to be the i-th eigenvalue of the Laplacian matrix $\mathcal{G}$ and the coupling strength, respectively, for two mutually coupled nodes then the mode i of Eq. (A.3) for a network with a number N of nodes will preserve the form of the mode i in Eq. (A.3) for the network with $N = 2$ if $\sigma(N) = 2\sigma(N = 2)/|\gamma_i(N)|$. For practical purposes, this relation can be expressed in terms of only the coupling strengths. Denoting $\tilde{\sigma}$ as the strength value for the linear coupling for which $\lambda^{(2)}(N = 2)$ reaches a given value, then the coupling strengths for which $\lambda^{(i)}(N)$ reaches the same value is given by the rescaling [18]

$$\tilde{\sigma}(N) = \frac{2\tilde{\sigma}(N = 2)}{|\gamma_i(N)|}. \tag{A.4}$$

A similar analysis can be done assuming that $\sigma = 0$. Once that $D_2\mathbf{S}(\mathbf{x}, \mathbf{x}) \ll D_1\mathbf{S}(\mathbf{x}, \mathbf{x})$ in Eq. (A.3), then the only term that changes in these equations as one considers networks with different topologies and sizes is $k(N)$, the number of connections a node within a network of N nodes receives from the other nodes. So, denoting $\tilde{g}$ as the strength values for the non-linear coupling for which $\lambda^{(2)}(N = 2)$ reaches a given value, then the coupling strength for which $\lambda^{(i)}(N)$ reaches the same value is given by the rescaling [18]

$$\tilde{g}(N) = \frac{\tilde{g}(N = 2)}{k}.$$

In this work we consider that Eqs. (6) and (7) remain valid if either $|\tilde{\sigma}/\tilde{g}| \gg 1$ or $|\tilde{g}/\tilde{\sigma}| \gg 1$, which means that one can consider the linear coupling as a perturbation ($|\tilde{g}/\tilde{\sigma}| \gg 1$) or the non-linear coupling as a perturbation ($|\tilde{\sigma}/\tilde{g}| \gg 1$). But, as shown in Ref. [18], these equations remain approximately valid even when $|\tilde{\sigma}/\tilde{g}| \approxeq 1$ or $|\tilde{g}/\tilde{\sigma}| \approxeq 1$.

Further in this work, the coupling interval is rescaled using as a reference the second largest conditional exponent $\lambda^{(2)}$ computed for the network with $N = 2$.

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